

Differential Equations Exam — Answer Key

1. Write the equation as $M dx + N dy = 0$ with $M = e^y + 4xy$, $N = xe^y + 2x^2 + 6y$. Check exactness:

$$\frac{\partial M}{\partial y} = e^y + 4x = \frac{\partial N}{\partial x}. \quad \checkmark \text{ exact}$$

Find $F(x, y)$ with $F_x = M$: $F = xe^y + 2x^2y + g(y)$. Then $F_y = xe^y + 2x^2 + g'(y) = N = xe^y + 2x^2 + 6y \Rightarrow g'(y) = 6y \Rightarrow g(y) = 3y^2$. So

$$F(x, y) = xe^y + 2x^2y + 3y^2 = C.$$

Applying $(x, y) = (2, 0)$: $2e^0 + 2(4)(0) + 3(0)^2 = 2 = C$. The implicit solution is

$$xe^y + 2x^2y + 3y^2 = 2.$$

2. Rewrite in standard form: $y' - \frac{4}{x}y = 3x^5$. Integrating factor: $\mu(x) = e^{\int -4/x dx} = x^{-4}$. Then

$$(x^{-4}y)' = 3x^5 \cdot x^{-4} = 3x \implies x^{-4}y = \frac{3}{2}x^2 + C \implies y = \frac{3}{2}x^6 + Cx^4.$$

Applying $y(1) = 5$: $\frac{3}{2} + C = 5 \Rightarrow C = \frac{7}{2}$. So

$$y(x) = \frac{3}{2}x^6 + \frac{7}{2}x^4.$$

3. The characteristic equation is $r^5 - 6r^4 + 10r^3 = 0$, i.e. $r^3(r^2 - 6r + 10) = 0$. This gives $r = 0$ (multiplicity 3) and $r^2 - 6r + 10 = 0 \Rightarrow r = \frac{6 \pm \sqrt{36 - 40}}{2} = 3 \pm i$. So the general solution is

$$y(x) = C_1 + C_2x + C_3x^2 + e^{3x}(C_4 \cos x + C_5 \sin x).$$

4. (a) Substituting the two data points into $P' = kP(M - P) - h$ with $h = 10$:

$$P = 4: \quad 4k(M - 4) - 10 = 6 \Rightarrow 4k(M - 4) = 16 \Rightarrow kM - 4k = 4.$$

$$P = 6: \quad 6k(M - 6) - 10 = 8 \Rightarrow 6k(M - 6) = 18 \Rightarrow kM - 6k = 3.$$

Subtracting, $2k = 1 \Rightarrow k = \frac{1}{2}$. Then $kM = 4k + 4 = 6 \Rightarrow M = 12$. So

$$P' = \frac{1}{2}P(12 - P) - 10.$$

(b) Critical points satisfy $\frac{1}{2}P(12 - P) - 10 = 0 \Rightarrow P(12 - P) = 20 \Rightarrow P^2 - 12P + 20 = 0 \Rightarrow (P - 2)(P - 10) = 0$, so $P = 2$ and $P = 10$.

Writing $f(P) = \frac{1}{2}P(12 - P) - 10 = -\frac{1}{2}P^2 + 6P - 10 = -\frac{1}{2}(P - 2)(P - 10)$, this is a downward-opening parabola in P that is negative for $P < 2$, positive for $2 < P < 10$, and negative for $P > 10$. On the phase line: arrows point down (decreasing) below $P = 2$, up between $P = 2$ and $P = 10$,

and down above $P = 10$. Hence $P = 2$ is an **unstable** critical point and $P = 10$ is a **stable** critical point — populations starting above the threshold $P = 2$ tend toward the sustainable equilibrium $P = 10$, while populations starting below 2 are driven to extinction.

5. Divide by x^2 : $y' = \frac{y}{x} + 4\left(\frac{y}{x}\right)^2$. This is homogeneous, so let $v = y/x$, i.e. $y = vx$, $y' = v + xv'$. Substituting,

$$v + xv' = v + 4v^2 \implies xv' = 4v^2 \implies \frac{dv}{v^2} = \frac{4dx}{x}.$$

Integrating: $-\frac{1}{v} = 4 \ln|x| + C$, so $v = \frac{-1}{4 \ln|x| + C}$. Since $y = vx$,

$$y(x) = \frac{-x}{4 \ln|x| + C}.$$

6. The characteristic equation is $r^3 + 9r = 0 \Rightarrow r(r^2 + 9) = 0 \Rightarrow r = 0, \pm 3i$, giving homogeneous solution $y_h = C_1 + C_2 \cos(3x) + C_3 \sin(3x)$.

For the forcing term $18x$: since $r = 0$ is a simple root of the characteristic equation, try $y_{p_1} = Ax^2 + Bx$. Then $y'_{p_1} = 2Ax + B$, $y'''_{p_1} = 0$, so

$$y'''_{p_1} + 9y'_{p_1} = 9(2Ax + B) = 18Ax + 9B = 18x \implies A = 1, B = 0 \implies y_{p_1} = x^2.$$

For the forcing term $26e^{2x}$: since $r = 2$ is not a root, try $y_{p_2} = Ce^{2x}$. Then

$$y'''_{p_2} + 9y'_{p_2} = 8Ce^{2x} + 18Ce^{2x} = 26Ce^{2x} = 26e^{2x} \implies C = 1 \implies y_{p_2} = e^{2x}.$$

The general solution is

$$y(x) = C_1 + C_2 \cos(3x) + C_3 \sin(3x) + x^2 + e^{2x}.$$