

Differential Equations Exam — Answer Key

1. First determine the external flows using the fact that each tank's volume is constant. Tank 1: inflow = 10(from T2)+ext_{in}, outflow = 20(to T2) ⇒ ext_{in,1} = 10. Tank 3: inflow = 40(from T2)+ext_{in}, outflow = 60(to T2) ⇒ ext_{in,3} = 20. Tank 2: inflow = 20 + 60 = 80, outflow = 10 + 40 + ext_{out} = 50 + ext_{out} ⇒ ext_{out,2} = 30.

Then

$$\begin{aligned}x'_1 &= 10(6) + 10 \cdot \frac{x_2}{200} - 20 \cdot \frac{x_1}{100} = 60 - \frac{1}{5}x_1 + \frac{1}{20}x_2, \\x'_2 &= 20 \cdot \frac{x_1}{100} + 60 \cdot \frac{x_3}{150} - (10 + 40 + 30) \cdot \frac{x_2}{200} = \frac{1}{5}x_1 - \frac{2}{5}x_2 + \frac{2}{5}x_3, \\x'_3 &= 40 \cdot \frac{x_2}{200} + 20(4) - 60 \cdot \frac{x_3}{150} = \frac{1}{5}x_2 + 80 - \frac{2}{5}x_3.\end{aligned}$$

So

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{20} & 0 \\ \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ 0 & \frac{1}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 60 \\ 0 \\ 80 \end{pmatrix}.$$

2. (a) Let $x_1 = x$, $x_2 = x'$. Then $x'_1 = x_2$ and $x'_2 = x'' = -t^2x_1 + e^{2t}$, with $x_1(0) = 2$, $x_2(0) = 3$. So

$$x'_1 = x_2, \quad x'_2 = -t^2x_1 + e^{2t}, \quad x_1(0) = 2, \quad x_2(0) = 3.$$

(b) From the second equation, $y' = \frac{x}{3}$. Differentiating the first equation, $x''' = 3y' + 4t^3$. Substituting $y' = x/3$:

$$x''' = 3 \cdot \frac{x}{3} + 4t^3 = x + 4t^3 \implies x''' - x = 4t^3.$$

3. $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$. Trace = -3, determinant = -4 + 6 = 2. Characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda + 1)(\lambda + 2) = 0 \Rightarrow \lambda = -1, -2$.

For $\lambda = -1$: $(A + I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \mathbf{v} = 0 \Rightarrow v_1 = v_2$; eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -2$: $(A + 2I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \mathbf{v} = 0 \Rightarrow 3v_1 = 2v_2$; eigenvector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

General solution: $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Applying $x(0) = 2$: $C_1 + 2C_2 = 2$. Applying $y(0) = 1$: $C_1 + 3C_2 = 1$. Subtracting, $C_2 = -1$, so $C_1 = 4$.

$$x(t) = 4e^{-t} - 2e^{-2t}, \quad y(t) = 4e^{-t} - 3e^{-2t}.$$

4. (a) Since trace(A) = $\lambda_1 + \lambda_2$: $2 + 3 = 5 = 6 + \lambda_2 \Rightarrow \lambda_2 = -1$.

(b) $(A + I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} \mathbf{v} = 0 \Rightarrow v_1 = -v_2$; eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(c) $\mathbf{x}(t) = C_1 e^{6t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

5. From Problem 3, $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$ has eigenvalues $-1, -2$ with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, giving fundamental matrix $\Phi(t) = \begin{pmatrix} e^{-t} & 2e^{-2t} \\ e^{-t} & 3e^{-2t} \end{pmatrix}$.

Since $\Phi(0) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ has determinant 1, $\Phi(0)^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$. So

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} e^{-t} & 2e^{-2t} \\ e^{-t} & 3e^{-2t} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3e^{-t} - 2e^{-2t} & 2e^{-2t} - 2e^{-t} \\ 3e^{-t} - 3e^{-2t} & 3e^{-2t} - 2e^{-t} \end{pmatrix}.$$

6. From $\mathbf{x}_c$, the fundamental matrix is $\Phi(t) = \begin{pmatrix} e^{2t} & e^{-3t} \\ 0 & 5e^{-3t} \end{pmatrix}$, with $\det \Phi(t) = 5e^{-t}$, so

$$\Phi(t)^{-1} = \begin{pmatrix} e^{-2t} & -\frac{1}{5}e^{-2t} \\ 0 & \frac{1}{5}e^{3t} \end{pmatrix}.$$

Then

$$\Phi(t)^{-1}\mathbf{f}(t) = \begin{pmatrix} e^{-2t} & -\frac{1}{5}e^{-2t} \\ 0 & \frac{1}{5}e^{3t} \end{pmatrix} \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Integrating (taking constants of integration to be zero, since we only need a particular solution):

$\int \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = \begin{pmatrix} t \\ 0 \end{pmatrix}$. So

$$\mathbf{x}_p = \Phi(t) \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t} & e^{-3t} \\ 0 & 5e^{-3t} \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} te^{2t} \\ 0 \end{pmatrix}.$$