

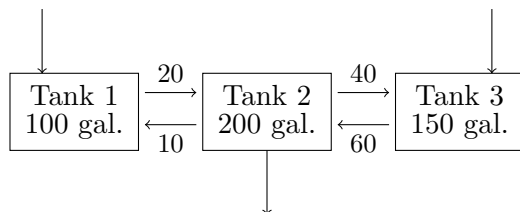
Differential Equations Exam

Show all work.

- There are three tanks with syrup (=sugar water) which satisfy the following (also see diagram):
 - Tank 1 has 100 gallons of syrup, Tank 2 has 200 gallons of syrup, and Tank 3 has 150 gallons of syrup. *The amount of syrup in each tank does not change.*
 - Tank 1 sends 20 gallons per minute into Tank 2; Tank 2 sends 10 gallons per minute into Tank 1 and also sends 40 gallons per minute into Tank 3; Tank 3 sends 60 gallons per minute into Tank 2. There are no other connections between tanks.
 - The concentration of sugar in the syrup coming into Tank 1 is 6 ounces per gallon. The concentration of sugar in the syrup coming into Tank 3 is 4 ounces per gallon.

Given that $x_1(t)$ is the amount of sugar in ounces in Tank 1 at time t , that $x_2(t)$ is the amount of sugar in ounces in Tank 2 at time t , and that $x_3(t)$ is the amount of sugar in ounces in Tank 3 at time t , then **set up the linear nonhomogeneous differential equation system that corresponds to this setup in matrix form.** Give your answer by putting numbers into the boxes below. *All entries must be whole numbers or reduced fractions.*

Hint: As a first step determine the rates of the external in/out flows in each tank.



$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \left(\begin{array}{c|c|c} \square & \square & 0 \\ \hline \square & \square & \square \\ \hline 0 & \square & \square \end{array} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

2. (a) Given the initial value problem $x'' + t^2x = e^{2t}$, with $x(0) = 2$ and $x'(0) = 3$, then give an equivalent first-order system with two equations, *including initial values*.

(b) Consider the linear system

$$\begin{aligned}x'' &= 3y + t^4 \\ 3y' - x &= 0.\end{aligned}$$

Find a single differential equation x satisfies (the answer should not involve y).

3. Solve for $x(t)$ and $y(t)$ given the following:

$$\begin{aligned}x' &= x - 2y & x(0) &= 2 \\y' &= 3x - 4y & y(0) &= 1\end{aligned}$$

4. Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$. Given that $\lambda_1 = 6$ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then carry out the following.

(a) Compute the other eigenvalue, λ_2 , of A .

(b) Find an eigenvector, $\mathbf{v}_2$, of A corresponding to the eigenvalue λ_2 .

(c) Write down the general form for solutions to the system $\mathbf{x}' = A\mathbf{x}$.

5. Compute e^{At} for $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$.

6. Given that $\mathbf{x}_c = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ is the general solution for the system $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix} \mathbf{x}$, then find a particular solution, $\mathbf{x}_p$, for the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.$$

Hint: The method of variation of parameters, $\mathbf{x}_p = \Phi(t) \int (\Phi(t))^{-1} \mathbf{f}(t) dt$, is more efficient than the method of undetermined coefficients for this problem.