

Differential Equations Exam — Answer Key

1. Let $x_1(t)$ and $x_2(t)$ be the amount of salt (oz) in Tank 1 and Tank 2. The volumes $V_1 = 100$ and $V_2 = 150$ are constant. Tank 1 receives 8 gal/min of fresh brine at 5 oz/gal, sends 16 gal/min to Tank 2, and receives 8 gal/min back from Tank 2. Tank 2 receives 16 gal/min from Tank 1 and sends 8 gal/min to Tank 1 and 8 gal/min out of the system. So

$$x_1' = 8(5) + 8 \cdot \frac{x_2}{150} - 16 \cdot \frac{x_1}{100} = 40 - \frac{4}{25}x_1 + \frac{4}{75}x_2,$$

$$x_2' = 16 \cdot \frac{x_1}{100} - (8 + 8) \cdot \frac{x_2}{150} = \frac{4}{25}x_1 - \frac{8}{75}x_2.$$

So the system is

$$\mathbf{x}' = \begin{pmatrix} -\frac{4}{25} & \frac{4}{75} \\ \frac{4}{25} & -\frac{8}{75} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 40 \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 20 \\ 25 \end{pmatrix}.$$

2. Let $x_1 = x$ and $x_2 = x'$. Then $x_1' = x_2$ and $x_2' = x'' = 3t^2x_1 + t^3x_2 + \sin(t)$. So

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ 3t^2 & t^3 \end{pmatrix}, \quad \vec{f}(t) = \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}.$$

3. From the first equation, $x' = x + 2y - e^t$, so $y = \frac{x' - x + e^t}{2}$. Differentiating, $y' = \frac{x'' - x' + e^t}{2}$. Substituting both into the second equation $y' = 3x + 2y + e^t$:

$$\frac{x'' - x' + e^t}{2} = 3x + 2 \cdot \frac{x' - x + e^t}{2} + e^t = 3x + (x' - x + e^t) + e^t = x' + 2x + 2e^t.$$

Multiplying both sides by 2:

$$x'' - x' + e^t = 2x' + 4x + 4e^t \implies x'' - 3x' - 4x = 3e^t.$$

4. $A = \begin{pmatrix} 0 & -4 \\ 4 & 8 \end{pmatrix}$. Trace = 8, determinant = $0(8) - (-4)(4) = 16$. Characteristic equation: $\lambda^2 - 8\lambda + 16 = 0 \implies (\lambda - 4)^2 = 0$, a repeated eigenvalue $\lambda = 4$. Since $(A - 4I) = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}$ satisfies $(A - 4I)^2 = 0$ (check: the rows of $A - 4I$ are proportional, so its square vanishes), we may use

$$e^{tA} = e^{4t}[I + t(A - 4I)] = e^{4t} \begin{pmatrix} 1 - 4t & -4t \\ 4t & 1 + 4t \end{pmatrix}.$$

Then

$$\mathbf{x}(t) = e^{tA}\mathbf{x}(0) = e^{4t} \begin{pmatrix} 1 - 4t & -4t \\ 4t & 1 + 4t \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = e^{4t} \begin{pmatrix} (1 - 4t) + 8t \\ 4t - 2(1 + 4t) \end{pmatrix} = \begin{pmatrix} (1 + 4t)e^{4t} \\ -(2 + 4t)e^{4t} \end{pmatrix}.$$

5. $A = \begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix}$.

(a) For $\mathbf{x}_1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$: $\mathbf{x}'_1 = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \end{bmatrix}$, and

$$A\mathbf{x}_1 = \begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix} \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 4e^{3t} - e^{3t} \\ 7e^{3t} - 4e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \mathbf{x}'_1. \checkmark$$

For $\mathbf{x}_2(t) = \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \end{bmatrix}$: $\mathbf{x}'_2 = \begin{bmatrix} -3e^{-3t} \\ -21e^{-3t} \end{bmatrix}$, and

$$A\mathbf{x}_2 = \begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix} \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \end{bmatrix} = \begin{bmatrix} 4e^{-3t} - 7e^{-3t} \\ 7e^{-3t} - 28e^{-3t} \end{bmatrix} = \begin{bmatrix} -3e^{-3t} \\ -21e^{-3t} \end{bmatrix} = \mathbf{x}'_2. \checkmark$$

Both are verified solutions.

(b) The Wronskian is

$$W(t) = \det \begin{pmatrix} e^{3t} & e^{-3t} \\ e^{3t} & 7e^{-3t} \end{pmatrix} = 7e^{3t}e^{-3t} - e^{-3t}e^{3t} = 7 - 1 = 6.$$

Since $W(t) = 6 \neq 0$ for all t , the solutions are linearly independent.

6. Using the fundamental matrix $\Phi(t) = \begin{pmatrix} e^{3t} & e^{-3t} \\ e^{3t} & 7e^{-3t} \end{pmatrix}$ formed from the solutions in Problem 5, we have $e^{t\mathbf{A}} = \Phi(t)\Phi(0)^{-1}$. Since

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ 1 & 7 \end{pmatrix}, \quad \det \Phi(0) = 6, \quad \Phi(0)^{-1} = \frac{1}{6} \begin{pmatrix} 7 & -1 \\ -1 & 1 \end{pmatrix},$$

we get

$$e^{t\mathbf{A}} = \begin{pmatrix} e^{3t} & e^{-3t} \\ e^{3t} & 7e^{-3t} \end{pmatrix} \cdot \frac{1}{6} \begin{pmatrix} 7 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 7e^{3t} - e^{-3t} & e^{-3t} - e^{3t} \\ 7e^{3t} - 7e^{-3t} & 7e^{-3t} - e^{3t} \end{pmatrix}.$$