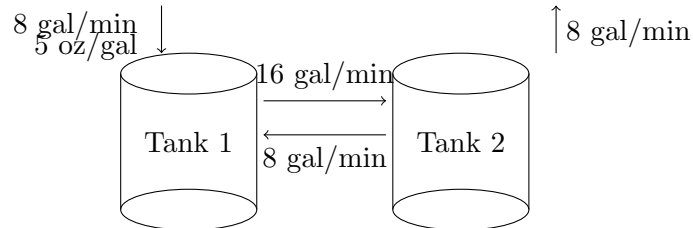


Differential Equations Exam

Show all work.

1. Two large tanks with brine solution (= salt water) are set up as illustrated in the following diagram.



Moreover, we have that initially (at $t = 0$), Tank 1 has 100 gallons of brine and 20 ounces of salt; while Tank 2 has 150 gallons of brine and 25 ounces of salt. Note that the flows are such that the volume of solution in each tank is constant.

Set up, *but do not solve*, an initial value problem of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}$$

with initial conditions $\mathbf{x}(0)$ that models the amount of salt in the two tanks.

2. Consider the second order differential equation

$$x'' - t^3 x' - 3t^2 x = \sin(t).$$

Transform it into an equivalent first order system of differential equations of the form

$$\frac{d}{dt}\vec{x} = A(t)\vec{x} + \vec{f}(t).$$

What are the vectors of unknown functions $\vec{x}$, the matrix of coefficients $A = A(t)$ and vector of functions $\vec{f}$ in the above system?

3. Consider the following linear system of equations

$$\begin{aligned}x' &= x + 2y - e^t \\ y' &= 3x + 2y + e^t\end{aligned}$$

Transform this system into a single differential equation in terms of x . Do not solve the equation.

4. Solve the following system using the matrix exponential.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -4 \\ 4 & 8 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

5. Consider the following first order linear system with constant coefficients:

$$\begin{cases} x' = 4x - y \\ y' = 7x - 4y. \end{cases}$$

(a) Verify that

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \end{bmatrix}$$

are solutions to the system.

(b) Compute the Wronskian of the two solutions from part a. Are the solutions linearly independent?

6. Compute the matrix exponential $e^{t\mathbf{A}}$ for the system in question 5. (*Hint: consider the solutions from part a of the problem*).