

Differential Equations Exam 4 — Answer Key

1. $y' = y^3 - 7y^2 + 10y = y(y - 2)(y - 5)$.

(a) Fixed points: $y = 0, 2, 5$.

(b) Sign of $f(y) = y(y - 2)(y - 5)$:

- $y < 0$: all three factors negative $\Rightarrow f(y) < 0$.
- $0 < y < 2$: $(+)(-)(-) \Rightarrow f(y) > 0$.
- $2 < y < 5$: $(+)(+)(-) \Rightarrow f(y) < 0$.
- $y > 5$: $(+)(+)(+) \Rightarrow f(y) > 0$.

So $y = 0$ is **unstable** ($- \rightarrow +$), $y = 2$ is **stable** ($+ \rightarrow -$), and $y = 5$ is **unstable** ($- \rightarrow +$). On the phase line: arrows point down below 0, up between 0 and 2, down between 2 and 5, and up above 5.

(c) The slope field has slope 0 along $y = 0, 2, 5$ (horizontal tangents), negative slope in the strips $y < 0$ and $2 < y < 5$, and positive slope in the strips $0 < y < 2$ and $y > 5$; solution curves flatten out and approach $y = 2$ from either side, and curve away from $y = 0$ and $y = 5$.

2. Let $y(t)$ be the pounds of salt in the tank at time t (minutes).

(a) Rate in = $(5 \text{ gal/min})(3 \text{ lb/gal}) = 15 \text{ lb/min}$. Rate out = $(5 \text{ gal/min})\left(\frac{y}{100} \text{ lb/gal}\right) = \frac{y}{20} \text{ lb/min}$. So

$$\frac{dy}{dt} = 15 - \frac{y}{20}, \quad y(0) = 20.$$

(b) Standard linear form: $y' + \frac{1}{20}y = 15$. Integrating factor $\mu(t) = e^{t/20}$. Then $(e^{t/20}y)' = 15e^{t/20}$, so $e^{t/20}y = 300e^{t/20} + C$, i.e.

$$y(t) = 300 + Ce^{-t/20}.$$

Applying $y(0) = 20$: $300 + C = 20 \Rightarrow C = -280$. So

$$y(t) = 300 - 280e^{-t/20}.$$

(c) As $t \rightarrow \infty$, $e^{-t/20} \rightarrow 0$, so $y(t) \rightarrow 300 \text{ lb}$. This is the equilibrium value at which the salt entering equals the salt leaving; physically, the outflow concentration $y(t)/100 \rightarrow 300/100 = 3 \text{ lb/gal}$, which matches the inflow concentration — once the tank is flushed out, its concentration matches what is coming in.

(d) Set $y(t) = 250$:

$$300 - 280e^{-t/20} = 250 \implies e^{-t/20} = \frac{50}{280} = \frac{5}{28} \implies t = 20 \ln\left(\frac{28}{5}\right) \approx 34.46 \text{ min}.$$

3.

(a) The equation $t^2y'' - 3ty' + y = \sin(t)$ is **second order** (highest derivative is y'') and **linear** (the dependent variable y and its derivatives appear only to the first power, with no products like yy').

(b) Yes, it is separable: multiply both sides by $ye^y dx$ to get

$$ye^y dy = (x^2 + 1) dx.$$

(c) Characteristic equation $r^2 + 6r + 9 = (r + 3)^2 = 0$ has a **repeated** root $r = -3$, so the system is **critically damped**.

(d)

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = e^{3t}(e^{3t} + 3te^{3t}) - te^{3t}(3e^{3t}) = e^{6t}.$$

Since $W = e^{6t} \neq 0$ for all t , $\{y_1, y_2\}$ is a fundamental set of solutions on $(-\infty, \infty)$.

(e) The origin fails to be the only equilibrium exactly when A is singular, i.e. $\det A = 0$:

$$\det A = 2k - (-1)(3) = 2k + 3 = 0 \implies k = -\frac{3}{2}.$$

At $k = -\frac{3}{2}$, A has a nontrivial null space, so every point on that eigenline (the eigenspace for eigenvalue 0) is also an equilibrium solution — the system has a whole line of equilibria rather than an isolated one.

4.

(a) Newton's second law: $mx'' = -kx + F(t)$, i.e. (with $m = 1$, $k = 25$)

$$x'' + 25x = 3 \cos(5t), \quad x(0) = 0, \quad x'(0) = 0.$$

(b) Homogeneous solution: $x_h = C_1 \cos(5t) + C_2 \sin(5t)$ (natural frequency $\omega_0 = \sqrt{25} = 5$). Since the forcing frequency (5) equals the natural frequency, this is **resonance**; try $x_p = t(A \cos 5t + B \sin 5t)$. Substituting (or using the standard resonance formula $x_p = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$ for forcing $F_0 \cos(\omega_0 t)$) gives

$$x_p(t) = \frac{3}{10} t \sin(5t).$$

General solution: $x(t) = C_1 \cos(5t) + C_2 \sin(5t) + \frac{3}{10} t \sin(5t)$. Applying $x(0) = 0$ gives $C_1 = 0$; applying $x'(0) = 0$ gives $C_2 = 0$. So

$$x(t) = \frac{3}{10} t \sin(5t).$$

(c) This is **pure resonance**: the driving frequency matches the system's natural frequency, so energy is continually pumped into the oscillation. The solution oscillates with period $2\pi/5$, but its amplitude $\frac{3}{10}t$ grows linearly without bound as $t \rightarrow \infty$ (rather than settling into a bounded steady state) — in a real spring this unbounded growth would eventually break the spring or leave the linear regime.

5.

(a) Salt leaves tank A at rate $5 \cdot \frac{x_1}{100} = \frac{x_1}{20}$ and enters from tank B at rate $5 \cdot \frac{x_2}{50} = \frac{x_2}{10}$, so $x_1' = -\frac{1}{20}x_1 + \frac{1}{10}x_2$. Symmetrically, $x_2' = \frac{1}{20}x_1 - \frac{1}{10}x_2$. In matrix form this is exactly $\vec{x}' = \begin{bmatrix} -\frac{1}{20} & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{10} \end{bmatrix} \vec{x}$.

(b) Trace $= -\frac{3}{20}$, determinant $= (-\frac{1}{20})(-\frac{1}{10}) - (\frac{1}{10})(\frac{1}{20}) = 0$. Characteristic equation $\lambda^2 + \frac{3}{20}\lambda = 0 \implies \lambda(\lambda + \frac{3}{20}) = 0 \implies \lambda = 0, -\frac{3}{20}$.

For $\lambda = 0$: $-\frac{1}{20}v_1 + \frac{1}{10}v_2 = 0 \Rightarrow v_1 = 2v_2$, eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For $\lambda = -\frac{3}{20}$: $\frac{1}{10}v_1 + \frac{1}{10}v_2 = 0 \Rightarrow v_1 = -v_2$, eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

General solution:

$$\vec{x}(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t/20}.$$

(c) $x_1(0) = 80$, $x_2(0) = 0$: $2C_1 + C_2 = 80$ and $C_1 - C_2 = 0 \Rightarrow C_1 = C_2$. So $3C_1 = 80 \Rightarrow C_1 = C_2 = \frac{80}{3}$. Thus

$$x_1(t) = \frac{160}{3} + \frac{80}{3}e^{-3t/20}, \quad x_2(t) = \frac{80}{3} - \frac{80}{3}e^{-3t/20}.$$

(d) The eigenvalue $\lambda = 0$ means the system does *not* decay to the origin; instead, as $t \rightarrow \infty$ the $e^{-3t/20}$ term vanishes and $\vec{x}(t) \rightarrow C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 160/3 \\ 80/3 \end{bmatrix} \approx \begin{bmatrix} 53.3 \\ 26.7 \end{bmatrix}$: the salt settles into the fixed 2:1 ratio matching the tanks' 100:50 volume ratio (equal final concentration ≈ 0.533 lb/gal in both tanks). Indeed,

$$x_1(t) + x_2(t) = \frac{160}{3} + \frac{80}{3}e^{-3t/20} + \frac{80}{3} - \frac{80}{3}e^{-3t/20} = \frac{240}{3} = 80$$

for all t — total salt is conserved, as it must be since no salt enters or leaves the two-tank system.

6. Taking the Laplace transform of $x'' + 4x = 2u_1(t) + 6\delta(t - 3)$ with $x(0) = x'(0) = 0$:

$$s^2X(s) + 4X(s) = 2 \cdot \frac{e^{-s}}{s} + 6e^{-3s} \implies X(s) = \frac{2e^{-s}}{s(s^2 + 4)} + \frac{6e^{-3s}}{s^2 + 4}.$$

Partial fractions: $\frac{1}{s(s^2 + 4)} = \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{4} \cdot \frac{s}{s^2 + 4}$, so

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s^2 + 4)}\right\} = \frac{1}{2} - \frac{1}{2}\cos(2t), \quad \mathcal{L}^{-1}\left\{\frac{6}{s^2 + 4}\right\} = 3\sin(2t).$$

Applying the second shifting theorem ($\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)u_a(t)$) to each term:

$$x(t) = \left[\frac{1}{2} - \frac{1}{2}\cos(2(t - 1))\right]u_1(t) + 3\sin(2(t - 3))u_3(t).$$

Written piecewise:

$$x(t) = \begin{cases} 0, & 0 \leq t < 1, \\ \frac{1}{2} - \frac{1}{2}\cos(2(t - 1)), & 1 \leq t < 3, \\ \frac{1}{2} - \frac{1}{2}\cos(2(t - 1)) + 3\sin(2(t - 3)), & t \geq 3. \end{cases}$$

The step force turns on smooth, bounded oscillation about the new equilibrium $x = \frac{1}{2}$ starting at $t = 1$; the impulse at $t = 3$ instantaneously increases the velocity by 6 (mass = 1), injecting an additional bounded oscillation $3\sin(2(t - 3))$ on top, with no resonance since $x(t)$ stays bounded for all t .

7. By definition,

$$\mathcal{L}\{t^3\} = \int_0^\infty t^3 e^{-st} dt =: I.$$

Integrate by parts with $u = t^3$, $dv = e^{-st} dt$ (so $v = -\frac{1}{s}e^{-st}$):

$$I = \left[-\frac{t^3}{s} e^{-st} \right]_0^\infty + \frac{3}{s} \int_0^\infty t^2 e^{-st} dt = \frac{3}{s} J_2,$$

where $J_2 = \int_0^\infty t^2 e^{-st} dt$. Integrate J_2 by parts with $u = t^2$, $dv = e^{-st} dt$:

$$J_2 = \left[-\frac{t^2}{s} e^{-st} \right]_0^\infty + \frac{2}{s} \int_0^\infty t e^{-st} dt = \frac{2}{s} J_1,$$

where $J_1 = \int_0^\infty t e^{-st} dt$. Integrate J_1 by parts with $u = t$, $dv = e^{-st} dt$:

$$J_1 = \left[-\frac{t}{s} e^{-st} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt = 0 + \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}.$$

Therefore

$$I = \mathcal{L}\{t^3\} = \frac{3}{s} \cdot \frac{2}{s} \cdot \frac{1}{s^2} = \frac{6}{s^4}. \quad \blacksquare$$