

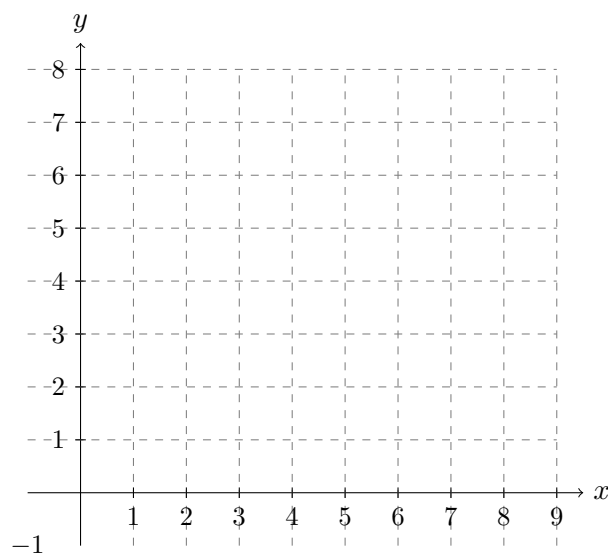
Differential Equations Exam

Show all work.

1. The following questions refer to the differential equation

$$y' = y^3 - 7y^2 + 10y.$$

- (a) Determine the fixed points of the differential equation.
- (b) Make a sketch of the phase diagram for the differential equation and indicate the stability of each of the fixed points.
- (c) Use the axes below to make a sketch of the slope field associated with the differential equation.



2. A 100-gallon tank is full of brine containing 20 lb of dissolved salt. Brine containing 3 lb of salt per gallon flows into the tank at a rate of 5 gal/min, and the well-stirred mixture flows out at the same rate, so the volume in the tank stays constant at 100 gallons.

(a) Let $y(t)$ be the number of pounds of salt in the tank at time t (in minutes). Set up an initial value problem for $y(t)$.

(b) Solve the initial value problem for $y(t)$.

- (c) What happens to the amount of salt in the tank as $t \rightarrow \infty$? Explain what this means physically in terms of the concentration of the outflow.
- (d) At what time t does the tank contain 250 lb of salt? Give an exact expression and a decimal approximation.

3. Answer each part below. **No full solution is required — justify each answer in a sentence or two, or with a short calculation.**

- (a) State the order of the differential equation $t^2y'' - 3ty' + y = \sin(t)$, and state whether it is linear or nonlinear.

- (b) Is the equation $\frac{dy}{dx} = \frac{x^2 + 1}{ye^y}$ separable? If so, rewrite it in separated form (you do not need to integrate).

- (c) Without solving, determine whether solutions of $x'' + 6x' + 9x = 0$ are overdamped, underdamped, or critically damped.

- (d) Compute the Wronskian of $y_1 = e^{3t}$ and $y_2 = te^{3t}$, and state whether $\{y_1, y_2\}$ is a fundamental set of solutions on $(-\infty, \infty)$.

- (e) The matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & k \end{bmatrix}$ arises in the system $\vec{x}' = A\vec{x}$. For all but one value of k , the origin is the *only* equilibrium solution. Find the exceptional value of k , and explain what is different about the system when k takes that value.

4. A 1-kg mass is attached to a spring with spring constant $k = 25$ N/m and rests at its equilibrium position on a frictionless horizontal track (no damping). Beginning at $t = 0$, an external force $F(t) = 3 \cos(5t)$ N is applied to the mass. The mass starts at rest at the equilibrium position.

(a) Using Newton's second law, write down the initial value problem satisfied by the displacement $x(t)$.

(b) Solve the initial value problem for $x(t)$.

- (c) What phenomenon occurs, and why? Describe the long-term behavior of $x(t)$, including how its amplitude changes with time and the period of oscillation.

5. Tank A holds 100 gallons of brine and tank B holds 50 gallons of brine. The wellstirred solution in tank A is pumped into tank B at 5 gal/min, and simultaneously the well-stirred solution in tank B is pumped into tank A at 5 gal/min, so both volumes stay constant. There is no flow into or out of the two-tank system. Let $x_1(t)$ and $x_2(t)$ be the pounds of salt in tanks A and B , respectively.

(a) Show that $\frac{d}{dt}\vec{x} = \begin{bmatrix} -\frac{1}{20} & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{10} \end{bmatrix} \vec{x}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of the coefficient matrix, and write down the general solution.

- (c) Suppose all 80 lb of salt starts in tank A , so $x_1(0) = 80$, $x_2(0) = 0$. Find the specific solution.
- (d) One of the eigenvalues you found in part (b) should be 0. Explain what this means for the long-term (as $t \rightarrow \infty$) distribution of salt between the two tanks, and verify that $x_1(t) + x_2(t)$ is constant for all t .

6. A 1-kg mass hangs on a spring with $k = 4$ N/m (undamped) and starts at rest at equilibrium: $x(0) = 0$, $x'(0) = 0$. Beginning at $t = 1$ s a constant force of 2 N is applied and left on; then at $t = 3$ s the mass additionally receives a sharp hammer blow that imparts an impulse of magnitude 6 (modeled as a Dirac delta function). That is,

$$x'' + 4x = 2u_1(t) + 6\delta(t - 3), \quad x(0) = 0, \quad x'(0) = 0,$$

where $u_1(t)$ is the unit step function that turns on at $t = 1$. Use the Laplace transform to solve for $x(t)$ for all $t \geq 0$.

7. Show that the Laplace transform of t^3 is equal to $\frac{6}{s^4}$ using the definition of the Laplace transform. (Do not use general properties of the transform but focus on this specific case. Show all of your work including each step.)