

Differential Equations Exam 5

Answer Key

1. The equation is $y' = y^3 - 5y^2 + 4y = y(y-1)(y-4)$.

(a) Setting $y' = 0$ gives $y(y-1)(y-4) = 0$, so the fixed points are

$$y = 0, \quad y = 1, \quad y = 4.$$

(b) Let $f(y) = y(y-1)(y-4)$ and test the sign of f on each interval:

interval	$y < 0$	$0 < y < 1$	$1 < y < 4$	$y > 4$
sign of $f(y)$	—	+	—	+

(e.g. $f(-1) = -10$, $f(0.5) = 0.875$, $f(2) = -4$, $f(5) = 20$.)

Since f changes $- \rightarrow +$ at $y = 0$, arrows point away on both sides: $y = 0$ is **unstable**. Since f changes $+ \rightarrow -$ at $y = 1$, arrows point toward it: $y = 1$ is **stable**. Since f changes $- \rightarrow +$ at $y = 4$, arrows point away: $y = 4$ is **unstable**.

Phase line: $\rightarrow 0 \leftarrow 1 \rightarrow 4 \rightarrow$

(c) Because the right-hand side depends only on y , all segments at a given height have the same slope, so the direction field is built from horizontal bands. **Zero slope** (flat segments, highlighted): along $y = 0$, $y = 1$, $y = 4$. **Positive slope** (segments tilt up): $0 < y < 1$ and $y > 4$. **Negative slope** (segments tilt down): $y < 0$ and $1 < y < 4$. Solutions starting between 0 and 4 (other than $y = 1$) approach $y = 1$; those starting above 4 or below 0 move away from $y = 4$ or $y = 0$.

2.

(a) Homogeneous: $2r^2 + 3r - 2 = 0 \Rightarrow r = \frac{-3 \pm 5}{4} = \frac{1}{2}, -2$, so $x_h = C_1 e^{t/2} + C_2 e^{-2t}$.

Since $\frac{1}{2}$ is already a root, try $x_p = Ate^{t/2}$. Substituting,

$$2x_p'' + 3x_p' - 2x_p = 5Ae^{t/2} \stackrel{!}{=} 5e^{t/2} \implies A = 1.$$

So $x(t) = C_1 e^{t/2} + C_2 e^{-2t} + te^{t/2}$.

Apply $x(0) = 0$: $C_1 + C_2 = 0$. Apply $x'(0) = 1$: $\frac{1}{2}C_1 - 2C_2 + 1 = 1 \Rightarrow \frac{1}{2}C_1 - 2C_2 = 0$. Using $C_2 = -C_1$: $\frac{1}{2}C_1 + 2C_1 = \frac{5}{2}C_1 = 0 \Rightarrow C_1 = 0$, $C_2 = 0$.

$$x(t) = te^{t/2}$$

This is a resonance case (the forcing $e^{t/2}$ matches the root $r = \frac{1}{2}$). Real, distinct roots $\Rightarrow$ no oscillations. The secular factor t means the solution grows faster than an ordinary exponential; the dominant term is $te^{t/2}$, which grows without bound.

(b) Homogeneous: $r^2 + 16 = 0 \Rightarrow x_h = C_1 \cos 4t + C_2 \sin 4t$. Since the forcing $\tan 4t$ is not a UC-eligible function, use variation of parameters with $y_1 = \cos 4t$, $y_2 = \sin 4t$, $W = 4$:

$$x_p = -y_1 \int \frac{y_2 \tan 4t}{4} dt + y_2 \int \frac{y_1 \tan 4t}{4} dt.$$

Using $\int \sin 4t \tan 4t dt = \frac{1}{4} \ln |\sec 4t + \tan 4t| - \frac{1}{4} \sin 4t$ and $\int \cos 4t \tan 4t dt = -\frac{1}{4} \cos 4t$, the $\sin 4t \cos 4t$ pieces cancel and

$$x(t) = C_1 \cos 4t + C_2 \sin 4t - \frac{1}{16} \cos 4t \ln |\sec 4t + \tan 4t|$$

The homogeneous part oscillates with period $2\pi/4 = \pi/2$ and constant amplitude. The particular term is an oscillation (period $\pi/2$) whose amplitude is modulated by $\ln |\sec 4t + \tan 4t|$, which grows without bound

as t approaches each vertical asymptote of $\tan 4t$ (at $t = \frac{\pi}{8} + \frac{k\pi}{4}$) — a resonance effect from forcing at the system's natural frequency. Dominant behavior: oscillation of period $\pi/2$ with unbounded (logarithmically growing) amplitude.

(c) $x' = e^{2x+5t} = e^{2x}e^{5t}$ is separable:

$$e^{-2x} dx = e^{5t} dt \implies -\frac{1}{2}e^{-2x} = \frac{1}{5}e^{5t} + C$$

$$e^{-2x} = -\frac{2}{5}e^{5t} + C_1 \implies \boxed{x(t) = -\frac{1}{2} \ln\left(C_1 - \frac{2}{5}e^{5t}\right)}$$

Since $x' = e^{2x+5t} > 0$ always, x increases monotonically (no oscillation). As t grows, the argument $C_1 - \frac{2}{5}e^{5t}$ decreases toward 0, so $x(t) \rightarrow +\infty$ in *finite time* $t^* = \frac{1}{5} \ln\left(\frac{5}{2}C_1\right)$ — a finite-time blow-up driven by the dominant term e^{5t} .

3. Write the equation as $M dx + N dy = 0$ with $M = 2xy + 1$, $N = x^2 + 3y^2$. Check exactness:

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}. \quad \checkmark \text{ exact}$$

Find $F(x, y)$ with $F_x = M$: $F = x^2y + x + g(y)$. Then $F_y = x^2 + g'(y) = N = x^2 + 3y^2 \implies g'(y) = 3y^2 \implies g(y) = y^3$. So

$$F(x, y) = x^2y + x + y^3 = C.$$

Applying $(x, y) = (1, 2)$: $1(2) + 1 + 2^3 = 2 + 1 + 8 = 11 = C$. The implicit solution is

$$x^2y + x + y^3 = 11.$$

4. Divide by x^2 : $y' = \frac{y}{x} + 5\left(\frac{y}{x}\right)^2$. This is homogeneous, so let $v = y/x$, i.e. $y = vx$, $y' = v + xv'$. Substituting,

$$v + xv' = v + 5v^2 \implies xv' = 5v^2 \implies \frac{dv}{v^2} = \frac{5 dx}{x}.$$

Integrating: $-\frac{1}{v} = 5 \ln|x| + C$, so $v = \frac{-1}{5 \ln|x| + C}$. Since $y = vx$,

$$y(x) = \frac{-x}{5 \ln|x| + C}.$$

5. Let $A = \begin{bmatrix} 0 & 1 \\ -13 & b \end{bmatrix}$. The characteristic equation is

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -13 & b - \lambda \end{bmatrix} = \lambda^2 - b\lambda + 13 = 0 \implies \lambda = \frac{b \pm \sqrt{b^2 - 52}}{2}.$$

So $\tau = \text{trace}(A) = b$ and $\det(A) = 0(b) - 1(-13) = 13$, which is **always positive**, regardless of b — so the origin is *never* a saddle here (that would require $\det < 0$). The discriminant $\Delta = b^2 - 52 = b^2 - (2\sqrt{13})^2$ splits the real line at $b = \pm 2\sqrt{13}$.

Rotation direction (same for every b). At the point $(x_1, x_2) = (1, 0)$ (positive x_1 -axis), the velocity is

$$\vec{x}' = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -13 \end{bmatrix},$$

which points straight down (in the $-x_2$ direction). A trajectory crossing the positive x_1 -axis moving downward is turning **clockwise**. Since this only depends on the off-diagonal entries 1, -13 (not on b), *every* spiral or center in this family rotates clockwise.

Regime 1: $b < -2\sqrt{13}$.

- $\tau = b < 0$, $\det = 13 > 0$.
- $\Delta = b^2 - 52 > 0$: two real, distinct, negative eigenvalues ($\lambda = \frac{b \pm \sqrt{\Delta}}{2}$, both negative since $|b| > \sqrt{\Delta}$).
- Geometric character: **stable node** (sink).
- Sketch: no rotation — trajectories approach the origin directly along two straight-line eigendirections, not spiraling.

Regime 2: $b = -2\sqrt{13}$.

- $\tau = -2\sqrt{13} < 0$, $\det = 13 > 0$, $\Delta = 0$.
- Repeated eigenvalue $\lambda = \tau/2 = -\sqrt{13}$ (double); solving $(A - \lambda I)\mathbf{v} = 0$ gives a single eigenvector $\mathbf{v} = (1, -\sqrt{13})$ (only one independent eigendirection, since A is not a scalar multiple of I).
- Geometric character: **stable degenerate (improper) node** — *not* a star node.
- Sketch: no rotation; all trajectories become tangent to the line through $(1, -\sqrt{13})$ as they approach the origin.

Regime 3: $-2\sqrt{13} < b < 0$.

- $\tau = b < 0$, $\det = 13 > 0$, $\Delta < 0$.
- Complex eigenvalues $\lambda = \frac{b}{2} \pm i \frac{\sqrt{52-b^2}}{2}$, negative real part.
- Geometric character: **stable spiral** (spiral sink).
- Sketch: trajectories spiral **clockwise** inward, winding onto the origin.

Regime 4: $b = 0$.

- $\tau = 0$, $\det = 13 > 0$, $\Delta = -52 < 0$.
- Purely imaginary eigenvalues $\lambda = \pm i\sqrt{13}$.
- Geometric character: **center**.
- Sketch: trajectories are closed elliptical orbits circulating **clockwise** around the origin, period $2\pi/\sqrt{13}$, neither growing nor decaying.

Regime 5: $0 < b < 2\sqrt{13}$.

- $\tau = b > 0$, $\det = 13 > 0$, $\Delta < 0$.
- Complex eigenvalues with positive real part.
- Geometric character: **unstable spiral** (spiral source).
- Sketch: trajectories spiral **clockwise** outward, away from the origin.

Regime 6: $b = 2\sqrt{13}$.

- $\tau = 2\sqrt{13} > 0$, $\det = 13 > 0$, $\Delta = 0$.
- Repeated eigenvalue $\lambda = \sqrt{13}$ (double); single eigenvector $\mathbf{v} = (1, \sqrt{13})$.
- Geometric character: **unstable degenerate (improper) node**.
- Sketch: no rotation; trajectories leave the origin tangent to the line through $(1, \sqrt{13})$.

Regime 7: $b > 2\sqrt{13}$.

- $\tau = b > 0$, $\det = 13 > 0$, $\Delta > 0$.
- Two real, distinct, positive eigenvalues.
- Geometric character: **unstable node** (source).
- Sketch: no rotation — trajectories leave the origin directly along two straight-line eigendirections.

Summary as b increases from $-\infty$ to $+\infty$: (stable node) $\xrightarrow{b=-2\sqrt{13}}$ (stable degenerate node) $\longrightarrow$ (stable spiral) $\xrightarrow{b=0}$ (center) $\longrightarrow$ (unstable spiral) $\xrightarrow{b=2\sqrt{13}}$ (unstable degenerate node) $\longrightarrow$ (unstable node). Because $\det(A) = 13 > 0$ for every b , a saddle never occurs; and because the off-diagonal entries never change sign, every rotating regime (spiral or center) turns **clockwise**.

6. Since the resistors are in series, $R = R_1 + R_2 = 2 + 4 = 6 \, \Omega$. For a series RC circuit, KVL gives $Ri + \frac{q}{C} = v(t)$ with $i = q'$, so with $R = 6$, $C = \frac{1}{12}$, and the initial charge $q(0) = 1$:

$$6q' + 12q = v(t), \quad q(0) = 1.$$

Write $v(t) = (6 - 6e^{-3t}) + u(t-2)(6e^{-3t} - 3)$ (one checks this equals $6 - 6e^{-3t}$ for $t < 2$ and 3 for $t \geq 2$).

Shifting so the bracket is a function of $t - 2$: $6e^{-3t} - 3 = 6e^{-6}e^{-3(t-2)} - 3$, so

$$V(s) = \frac{6}{s} - \frac{6}{s+3} + e^{-2s} \left(\frac{6e^{-6}}{s+3} - \frac{3}{s} \right).$$

Transforming the ODE, with $\mathcal{L}\{q'\} = sQ(s) - q(0) = sQ(s) - 1$:

$$6(sQ(s) - 1) + 12Q(s) = V(s) \Rightarrow Q(s) = \frac{1}{s+2} + \frac{V(s)}{6(s+2)}.$$

The first term (from the initial charge) inverts to e^{-2t} . For the forcing term, use partial fractions $\frac{1}{s(s+2)} = \frac{1/2}{s} - \frac{1/2}{s+2}$ and $\frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$; the un-shifted piece

$$\frac{1}{6(s+2)} \left(\frac{6}{s} - \frac{6}{s+3} \right) = \frac{1}{s(s+2)} - \frac{1}{(s+2)(s+3)}$$

inverts to

$$q_1(t) = \frac{1}{2} - \frac{3}{2}e^{-2t} + e^{-3t}.$$

For the shifted piece, with $H(s) = \frac{6e^{-6}}{6(s+2)(s+3)} - \frac{3}{6s(s+2)} = \frac{e^{-6}}{(s+2)(s+3)} - \frac{1/2}{s(s+2)}$,

$$h(\tau) = e^{-6}e^{-2\tau} - e^{-6}e^{-3\tau} - \frac{1}{4} + \frac{1}{4}e^{-2\tau},$$

and by the second shifting theorem the extra contribution for $t \geq 2$ is $h(t-2)$.

Now combine $e^{-2t} + q_1(t)$ (and $h(t-2)$ for $t \geq 2$). For $0 \leq t < 2$ the e^{-2t} terms combine as $-\frac{3}{2}e^{-2t} + e^{-2t} = -\frac{1}{2}e^{-2t}$. For $t \geq 2$, use $e^{-6}e^{-3(t-2)} = e^{-3t}$, so the e^{-3t} terms in q_1 and $h(t-2)$ cancel, and write $e^{-2t} = e^{-4}e^{-2(t-2)}$, leaving a single natural-mode exponential $e^{-2(t-2)}$:

$$q(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t} + e^{-3t}, & 0 \leq t < 2, \\ \frac{1}{4} + \left(\frac{1}{4} - \frac{1}{2}e^{-4} + e^{-6} \right) e^{-2(t-2)}, & t \geq 2. \end{cases}$$

(Numerically, $q(t) \approx 0.25 + 0.2433e^{-2(t-2)}$ for $t \geq 2$.) One can check that $q(0) = \frac{1}{2} - \frac{1}{2} + 1 = 1$, that $q(t)$ is continuous at $t = 2$ (both pieces give $q(2) \approx 0.4933$), and that $q(t) \rightarrow \frac{1}{4}$ as $t \rightarrow \infty$, matching the steady charge $v(\infty)/(1/C) = 3/12 = 1/4$.

7. By definition,

$$\mathcal{L}\{t^{3/2}\} = \int_0^\infty e^{-st} t^{3/2} dt.$$

Step 1 (integrate by parts): let $u = t^{3/2}$, $dv = e^{-st} dt$, so $du = \frac{3}{2}t^{1/2}dt$, $v = -\frac{1}{s}e^{-st}$. The boundary term $-\frac{1}{s}t^{3/2}e^{-st}\big|_0^\infty$ vanishes at both ends ($t = 0$ kills $t^{3/2}$; $t \rightarrow \infty$, e^{-st} dominates any power of t for $s > 0$), leaving

$$\int_0^\infty e^{-st} t^{3/2} dt = \frac{3}{2s} \int_0^\infty e^{-st} t^{1/2} dt.$$

Step 2 (integrate by parts again): with $u = t^{1/2}$, $dv = e^{-st} dt$, $du = \frac{1}{2}t^{-1/2}dt$, $v = -\frac{1}{s}e^{-st}$; the boundary term again vanishes, giving

$$\int_0^\infty e^{-st} t^{1/2} dt = \frac{1}{2s} \int_0^\infty e^{-st} t^{-1/2} dt.$$

Step 3 (the base integral): substitute $u = st$ ($t = u/s$, $dt = du/s$) in $K = \int_0^\infty e^{-st} t^{-1/2} dt$:

$$K = \int_0^\infty e^{-u} \left(\frac{u}{s} \right)^{-1/2} \frac{du}{s} = \frac{1}{\sqrt{s}} \int_0^\infty u^{-1/2} e^{-u} du.$$

Evaluate $\int_0^\infty u^{-1/2} e^{-u} du$ by setting $u = w^2$ ($du = 2w dw$):

$$\int_0^\infty u^{-1/2} e^{-u} du = \int_0^\infty w^{-1} e^{-w^2} (2w) dw = 2 \int_0^\infty e^{-w^2} dw = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi},$$

using the classical Gaussian integral $\int_0^\infty e^{-w^2} dw = \sqrt{\pi}/2$. Hence $K = \sqrt{\pi/s}$.

Step 4 (assemble):

$$\begin{aligned} \int_0^\infty e^{-st} t^{1/2} dt &= \frac{1}{2s} \sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}, \\ \int_0^\infty e^{-st} t^{3/2} dt &= \frac{3}{2s} \cdot \frac{\sqrt{\pi}}{2s^{3/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}. \end{aligned}$$

$$\boxed{\mathcal{L}\{t^{3/2}\} = \frac{3\sqrt{\pi}}{4s^{5/2}}} \quad \blacksquare$$