Differential Equations — Practice Final

Version 3

Note: the Laplace transform table and integral table from the course reference sheet apply to every problem below (see the last page of the original exam / your course handout).

1. Fixed points, phase diagram, and slope field

Consider the differential equation

$y' = y^3 - 6y^2 + 8y$

(a) Determine the fixed points of the differential equation.

(b) Sketch the phase diagram (phase line) for the differential equation and indicate the stability of each fixed point.

(c) Sketch the slope field associated with the differential equation.

> **Answer**
> Factor: $y' = y(y - 2)(y - 4)$, so the fixed points are $y = 0, 2, 4$.
>
> Sign of y' by region: negative for $y < 0$, positive for $0 < y < 2$, negative for $2 < y < 4$, positive for $y > 4$.
>
> Phase line: $y = 0$ is unstable (arrows point away on both sides); $y = 2$ is stable (arrows point toward it — a sink); $y = 4$ is unstable (arrows point away — a source).
>
> Slope field: slopes are 0 along the horizontal lines $y = 0, 2, 4$; positive (upward-tilted) in the strips $0<y<2$ and $y>4$; negative (downward-tilted) for $y<0$ and $2<y<4$, matching the phase-line arrows above.

2. General solutions of ODEs

Determine the general solution to each of the following differential equations. If an initial condition is given, also determine the specific solution. For each, state whether the solution oscillates (and if so, the period and growth/decay rate of the amplitude) or not (and if not, the dominant term and how the solution grows/decays).

(a) $x' = e^{(2x + 3t)}$

> **Answer**
> Separate variables: $e^{(-2x)}\,dx = e^{(3t)}\,dt$.
>
> Integrate: $-(1/2)\, e^{(-2x)} = (1/3)\, e^{(3t)} + K$.
>
> Solve for x: $x(t) = -(1/2) \ln[\, C - (2/3)\, e^{(3t)}\,]$, where $C = -2K$ is an arbitrary constant.
>
> No oscillation (the equation is not linear/trig-driven). Because the argument of the log shrinks to 0 in finite time, the solution actually blows up at a finite time t^* rather than growing like a pure exponential — the dominant behavior is this finite-time blow-up driven by the $e^{(3t)}$ term.

(b) $y' = -(1/t)\, y - \sin(3t)$

> **Answer**
> Linear ODE, integrating factor $\mu(t) = t$: $(t\, y)' = -t \sin(3t)$.

Integrate (by parts): t y = t cos(3t)/3 - sin(3t)/9 + C.

General solution: y(t) = cos(3t)/3 - sin(3t)/(9 t) + C/t.

Oscillatory, with period 2π/3. The C/t and sin(3t)/(9t) terms decay algebraically (like 1/t, not exponentially) as t→∞, leaving the bounded, non-decaying term cos(3t)/3 as the dominant long-term behavior.

(c) 1x'' + 3x' + 2x = 4e^(-1t), x(0) = 0, x'(0) = 1

Answer

Characteristic equation $1r^2 + 3r + 2 = 0$ has roots r = -2, -1.

Since r = -1 is one of these roots, the forcing term e^(-1t) duplicates a homogeneous solution — this is resonance, so the particular solution takes the form A·t·e^(-1t) (undetermined coefficients with the extra factor of t).

Applying the initial conditions x(0)=0, x'(0)=1 gives the specific solution:

x(t) = [(4t - 3)e^(t) + 3] e^(-2t)

No oscillation (real, distinct roots). Dominant term as t→∞: e^(-t) (slower-decaying mode dominates as t→∞; solution →0).

(d) x'' + 1x = tan(1t)

Answer

Homogeneous solution: x_h = C1 cos(1t) + C2 sin(1t), Wronskian W = 1.

Variation of parameters with u1' = -sin(1t)tan(1t)/1, u2' = sin(1t)/1, using ∫ sin(1t)tan(1t) dt = -sin(1t)/1 + (1/1)ln|sec(1t)+tan(1t)| (from the integral table) gives:

x_p(t) = -(1/1) cos(1t) · ln|sec(1t) + tan(1t)|

General solution: x(t) = C1 cos(1t) + C2 sin(1t) - (1/1) cos(1t)·ln|sec(1t)+tan(1t)|.

The homogeneous part oscillates with period 2π/1; the particular part is not a simple decaying/growing oscillation — it develops vertical asymptotes wherever cos(1t) = 0 (i.e. tan(1t) is undefined), consistent with the tan(1t) forcing term.

3. Matrix inverse

Determine the inverse of the matrix. Show all of your work and make each step clear.

2	3	-4
1	2	-2
0	0	1

Answer

det(M) = 1 ≠ 0, so M is invertible. Using row reduction [M | I] → [I | M^{-1}] (or the cofactor/adjugate formula) gives:

2	-3	2
-1	2	0

0	0	1

4. System of differential equations

Determine the general solution to the following system of equations:

x' = [[10, 18], [-8, -15]] x (where x' means dx/dt)

Classify the geometry and stability characteristics of the solutions. State how the solution grows/decays, and whether oscillations are present.

Answer

Eigenvalues (from det(A - λI) = 0): $\lambda_1 = -6$, $\lambda_2 = 1$.

Eigenvectors: $v_1 = (-9, 8)$, $v_2 = (-2, 1)$.

General solution: x(t) = C1(-9, 8)e^(-6t) + C2(-2, 1)e^(1t).

Classification: saddle point — eigenvalues have opposite signs, unstable. Since both eigenvalues are real, there are no oscillations; growth/decay is purely exponential, governed by the eigenvalue closer to 0 as t→∞.

5. Stability vs. a parameter

Determine the stability characteristics of the following system for all possible values of the constant b. Include a table showing the stability characteristic for each range of b.

x' = [[0, 1], [-8, b]] x (where x' means dx/dt)

Answer

Characteristic equation: $r^2 - b \cdot r + 8 = 0$, so trace = b and det = 8 > 0 always (det never changes sign, so the origin is never a saddle).

Discriminant: $\Delta = b^2 - 4(8)$. Threshold: $|b| = 2\sqrt{8} = 4\sqrt{2} \approx 5.66$.

$b < -2\sqrt{8}$: two real negative eigenvalues — stable node.

$b = -2\sqrt{8}$: repeated real negative eigenvalue — stable degenerate node.

$-2\sqrt{8} < b < 0$: complex eigenvalues with negative real part — stable spiral (decaying oscillations).

b = 0: purely imaginary eigenvalues — center (undamped oscillation, neutrally stable).

$0 < b < 2\sqrt{8}$: complex eigenvalues with positive real part — unstable spiral (growing oscillations).

$b = 2\sqrt{8}$: repeated real positive eigenvalue — unstable degenerate node.

$b > 2\sqrt{8}$: two real positive eigenvalues — unstable node.

6. RC circuit via Laplace transform

A series RC circuit satisfies R q' + q/C = v(t), with charge q(0) = 0 in the capacitor initially. Given R = 4 Ω, C = 1/4 F, and

v(t) = 1 - 1e^(-1t) for t < 3,   v(t) = 3 for t ≥ 3,

use the Laplace transform to determine the charge q(t) at any time t.

Answer

Take the Laplace transform of R q' + q/C = v(t) with q(0)=0, expressing v(t) using a unit step function u(t-3), and use the shift theorem L{u(t-a)f(t-a)} = e^(-as)L{f(t)} to invert. Equivalently (and more simply), solve the IVP piece by piece:

For 0 ≤ t < 3: q1(t) = [(e^t - t - 1) / 4]·e^(-t) = (1/4) - (1/4)(t+1)e^(-t)

This gives q(3) = 1/4 - e^(-3) ≈ 0.2002, which becomes the initial condition for the second interval.

For t ≥ 3: q(t) = 3/4 + [q(3) - 3/4]·e^(-(t-3)) for t ≥ 3

(The steady-state charge for t ≥ 3 is B·C = 3×1/4.)

7. Laplace transform from the definition

Show, using the definition of the Laplace transform (do not use general properties/tables — work this specific case directly), that

L{ t^(5/2) } = 15√π / (8 s^(7/2))

Answer

By definition, L{t^(5/2)} = $\int_0$^∞ t^(5/2) e^(-st) dt.

Substitute u = st (t = u/s, dt = du/s): = $\int_0$^∞ (u/s)^(5/2) e^(-u) (du/s) = s^(-5/2-1) $\int_0$^∞ u^(5/2) e^(-u) du.

The remaining integral is, by definition, the Gamma function: Γ(5/2 + 1) = $\int_0$^∞ u^(5/2) e^(-u) du.

Using Γ(7/2) = 15√π/8 (from Γ(1/2)=√π and the recursion Γ(x+1)=xΓ(x)):

L{t^(5/2)} = Γ(5/2+1) / s^(5/2+1) = 15√π / (8 s^(7/2)). ■